PHYC 511

Spring 2019

Problem Session 1 01/16/2019

(1) Starting from the divergence and Stoke's theorem, prove the following relations:

(i) $\int_V \vec{\nabla}\psi \, d\tau = \oint_S \psi \hat{n} \, da$

(ii) $\int_V \vec{\nabla} \times \vec{A} \, d\tau = \oint_S \hat{n} \times \vec{A} \, da$

(iii) $\int_S \hat{n} \times \vec{\nabla}\psi \, da = \oint_C \psi \, d\vec{l}$

(2) Problem 1.3, Jackson.

(4) (i) Starting with ψ, we can define:

$$\vec{A} = \psi \vec{i}$$

The divergence theorem states that:

$$\int_V \vec{\nabla} \cdot \vec{A} \, d\tau = \oint_S \vec{A} \cdot \hat{n} \, da \Rightarrow \int_V \frac{\partial \psi}{\partial x} d\tau = \oint_S \psi \, n_x \, da$$

We can repeat by defining $\vec{A} = \psi \vec{j}$ and $\vec{A} = \psi \vec{k}$, resulting in:

$$\int_V \frac{\partial \psi}{\partial y} d\tau = \oint_S \psi \, n_y \, da \quad , \quad \int_V \frac{\partial \psi}{\partial z} d\tau = \oint_S \psi \, n_z \, da$$

The three relations together imply that:

$$\int_V \vec{\nabla} \psi \, d\tau = \int_V \frac{\partial \psi}{\partial x} d\tau \, \vec{i} + \int_V \frac{\partial \psi}{\partial y} d\tau \, \vec{j} + \int_V \frac{\partial \psi}{\partial z} d\tau \, \vec{k} = \oint_S \psi \, n_x \, da \, \vec{i}$$

$$+ \oint_S \psi \, n_y \, da \, \vec{j} + \oint_S \psi \, n_z \, da \, \vec{k} = \oint_S \psi \, \hat{n} \, da$$

(ii) We can apply (i) by choosing $\psi = A_x$:

$$\int_V \vec{\nabla} A_x \, d\tau = \oint_S A_x \, \hat{n} \, da \Rightarrow \int_V \frac{\partial A_x}{\partial y} d\tau = \oint_S A_x \, n_y \, da \, ,$$

$$\int \frac{\partial A_x}{\partial z} d\tau = \oint_S A_x \, n_z \, da \quad (*)$$

Similarly, for $\psi = A_y$, (i) yields:

$$\int_V \vec{\nabla} A_y \, d\tau = \oint_S A_y \hat{n} \, da \Rightarrow \int_V \frac{\partial A_y}{\partial x} d\tau = \oint_S A_y n_x \, da ,$$

$$\int_V \frac{\partial A_y}{\partial z} d\tau = \oint_S A_y n_z \, da \qquad (**)$$

Finally, for $\psi = A_z$, we find from (i):

$$\int_V \vec{\nabla} A_z \, d\tau = \oint_S A_z \hat{n} \, da \Rightarrow \int_V \frac{\partial A_z}{\partial x} d\tau = \oint_S A_z n_x \, da ,$$

$$\int_V \frac{\partial A_z}{\partial y} d\tau = \oint_S A_z n_y \, da \qquad (***)$$

By subtracting the two sides of the second relation in $(**)$ from those of the second relation in $(***)$, we find:

$$\int_V \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) d\tau = \oint_S (n_y A_z - n_z A_y) \, da \qquad (a)$$

From the other four relations in $(*)$, $(**)$, $(***)$, we can also find:

$$\int_V \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) d\tau = \oint_S (n_z A_x - n_x A_z) \, da \qquad (b)$$

$$\int_V \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) d\tau = \oint_S (n_x A_y - n_y A_x) \, da \qquad (c)$$

Together, (a), (b), (c) give:

$$\int_V \vec{\nabla} \times \vec{A}\, d\tau = \oint_S \hat{n} \times \vec{A}\, da$$

(iii) The Stoke's theorem states that:

$$\oint_C \vec{A} \cdot d\vec{\ell} = \int_S (\vec{\nabla} \times \vec{A}) \cdot \hat{n}\, da$$

By choosing $\vec{A} = \psi\, \vec{i}$, this results in:

$$\oint_C \vec{A} \cdot d\vec{\ell} = \oint_C \psi\, d\ell_x$$

$$\vec{\nabla} \times \vec{A} = \frac{\partial \psi}{\partial z}\, \vec{j} - \frac{\partial \psi}{\partial y}\, \vec{k} \Rightarrow \int_S (\vec{\nabla} \times \vec{A}) \cdot \hat{n}\, da = \int_S \left(n_y \frac{\partial \psi}{\partial z} - n_z \frac{\partial \psi}{\partial y}\right) da$$

And hence:

$$\int_S \left(n_y \frac{\partial \psi}{\partial z} - n_z \frac{\partial \psi}{\partial y}\right) da = \oint_C \psi\, d\ell_x \qquad (d)$$

Similarly, by choosing $\vec{A} = \psi\, \vec{j}$, $\vec{A} = \psi\, \vec{k}$, we find from Stoke's theorem:

$$\int_S \left(n_z \frac{\partial \psi}{\partial x} - n_x \frac{\partial \psi}{\partial z}\right) da = \oint_C \psi\, d\ell_y \qquad (e)$$

$$\int_S \left(n_x \frac{\partial \psi}{\partial y} - n_y \frac{\partial \psi}{\partial x}\right) da = \oint_C \psi\, d\ell_z \qquad (f)$$

Together, (d), (e), (f) give:

$$\int_S \hat{n} \times \vec{\nabla} \psi \, da = \oint_C \psi \, d\vec{l}$$

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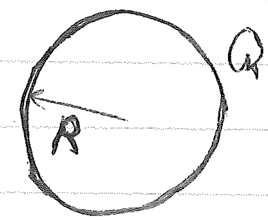
(2) (a) $\rho(\vec{x}) = \dfrac{Q}{4\pi R^2} \delta(r-R)$

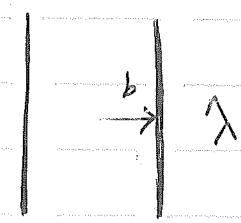


This satisfies:

integration over θ, ϕ

$$\int \rho(\vec{x}) d\tau = \int_0^\infty \frac{Q}{4\pi R^2} \delta(r-R) \, r^2 dr \times 4\pi = \boxed{Q}$$

(b) $\rho(\vec{x}) = \dfrac{\lambda}{2\pi b} \delta(\rho - b)$

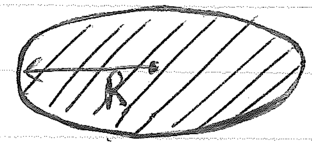


This satisfies:

integration over ϕ

$$\int \rho(\vec{x}) d\tau = \int_0^\infty \frac{\lambda}{2\pi b} \delta(\rho - b) \rho \, d\rho \times 2\pi = \boxed{\lambda}$$

(c) $\rho(\vec{x}) = \begin{cases} \dfrac{Q}{\pi R^2} \delta(z) & \rho < R \\ 0 & \rho > R \end{cases}$



This satisfies:

integration over ϕ

$$\int \rho(\vec{x}) d\tau = \int_{-\infty}^{+\infty} \frac{Q}{\pi R^2} \delta(z) \, dz \int_0^R \rho \, d\rho \times 2\pi = \boxed{Q}$$

(d) $\rho(\vec{x}) = \frac{C}{r} \delta\left(\theta - \frac{\pi}{2}\right) \Theta(R-r)$

$r \rightarrow$ in order to give the correct charge within radius r on the disk

This results in:

integration over ϕ

$$\int \rho(\vec{x})\, d\tau = C \int_0^\infty \Theta(R-r)\, r\, dr \int_0^\pi \delta\left(\theta - \frac{\pi}{2}\right) \sin\theta\, d\theta \times \hat{2\pi} = \pi R^2 C$$

needed

$= Q \Rightarrow C = \frac{Q}{\pi R^2}$

Thus:

$$\boxed{\rho(\vec{x}) = \frac{Q}{\pi R^2} \frac{1}{r} \delta\left(\theta - \frac{\pi}{2}\right) \Theta(R-r)}$$